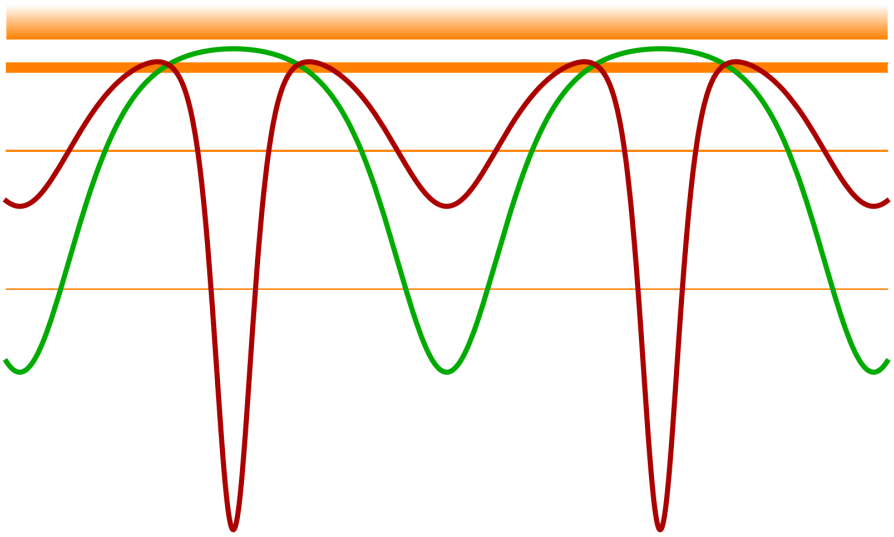
Institute for Theoretical Physics
Faculty of Science
University of Bern

Bachelor Thesis

on

Supersymmetric Quantum Mechanics



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Abstract

This bachelor thesis deals with nonrelativistic supersymmetric quantum mechanics in one dimension. We will discuss the formalism of Supersymmetry along with a few applications such as the infinite square well potential and the harmonic oscillator. Special attention will be given to the delta-well potential first as a single and thereafter periodically repeated potential. Ultimately, we will examine the Lamé potentials as a semi-analytically solvable class of periodic potentials and investigate their supersymmetric partner potentials.

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1 Motivation

In order to describe nature, symmetries play an important role in physics. When the laws describing a physical system are left invariant under certain transformations, the system has a symmetry. To be precise, one can distinguish between two different types of symmetries: For instance, a sphere is symmetrical under rotations by any angle – this symmetry belongs to the class of continuous symmetries. In contrast, a one-dimensional periodic potential (e.g. the Lamé potentials) is invariant under discrete translations by integer multiples of its lattice constant a – therefore it is called a discrete symmetry. At the beginning of the last century, H. Poincaré and E. Noether derived the strong relation between symmetries and corresponding conserved quantities.
Most symmetries lead to degeneracies in the spectrum of the quantum mechanical Hamiltonian, i.e. two or more states belong to each energy level. The idea of Supersymmetry (SUSY) was born in quantum field theories, it accordingly gives rise to the supersymmetric partners of all the particles in the mass resp. energy spectrum: the so-called SUSY particles (which among others are being searched for at the LHC at CERN, recently put into operation). Hereby, supersymmetric transformations change bosonic degrees of freedom (integer spin) into fermionic ones (half-integer spin) and vice versa. The SUSY transformations are carried out by a supersymmetric operator $\mathbb{Q}$

$$\mathbb{Q}\,|boson\rangle \propto |fermion\rangle \qquad \text{and} \qquad \mathbb{Q}\,|fermion\rangle \propto |boson\rangle \quad . \tag{1.1}$$

Supersymmetry has not yet been observed in nature. Therefore it must be broken in such a way that the supersymmetric partners are rather massive and thus difficult to produce in particle accelerators. Understanding of the breaking mechanism is difficult in the context of quantum field theories – this is how the idea of transferring SUSY into quantum mechanics arose, allowing a simpler approach. This approach will be presented and discussed throughout the following chapters. As we will soon see, it turned out that SUSY quantum mechanics has many interesting results itself.
This section is based on [3].

1.1 Group Theory

In the following, only a few essential facts regarding groups and algebras will be discussed. For further details one may consider the referenced literature.
In a mathematical context, symmetries can be expressed by groups: continuous symmetries are generally described by Lie groups and discrete symmetries by finite groups. Lie groups have an infinite number of elements but may be described by a finite set of parameters.
An algebra is a vector space on a group, whose basis consists of generators. By definition, it has a product defined between its generators such that the result is an element of the algebra again. In the case of a Lie algebra, this resembles the commutator between two operators

$$\mathbb{A} \circ \mathbb{B} \doteq [\mathbb{A},\ \mathbb{B}] \quad . \tag{1.2}$$

In Lie groups, finite transformations can be composed of infinitesimal transformations. The corresponding generators form a Lie algebra. For instance, the

angular momentum operators form a set of generators and their commutator relations $[\mathbb{L}_k,\ \mathbb{L}_l] = i\hbar\,\epsilon_{klm}\,\mathbb{L}_m$ realize the product of the Lie algebra.
The algebra of SUSY consists of additional anticommutators – thus it poses a generalization of a Lie algebra, a so-called Lie superalgebra (SUSY algebra)

$$\begin{aligned} [\mathbb{H}^{(S)},\ \mathbb{Q}_i] &\equiv \mathbb{H}^{(S)}\,\mathbb{Q}_i - \mathbb{Q}_i\,\mathbb{H}^{(S)} = 0 \quad , \\ \{\mathbb{Q}_i,\ \mathbb{Q}_k\} &\equiv \mathbb{Q}_i\,\mathbb{Q}_k + \mathbb{Q}_k\,\mathbb{Q}_i = \mathbb{H}^{(S)}\,\delta_{ik} \quad , \\ &\text{where } i,k = 1,...,N \quad . \end{aligned} \tag{1.3}$$

These relations together form the so-called N-extended SUSY algebra (in a quantum system described by the supersymmetric Hamiltonian $\mathbb{H}^{(S)}$, the superscript S means supersymmetric). Hereby, the already introduced SUSY transformation operators $\mathbb{Q}_i$ embody the algebra generators: they are called supercharges. SUSY models can be classified by the number of linearly independent, anticommuting generators.
In a quantum mechanical context, $N = 1$ extended SUSY only resembles a free particle Hamiltonian $\mathbb{H} = \frac{1}{2m}\,\mathbb{P}^2$. $N = 2$ extended SUSY is the first more interesting case and relates two different potentials by the two supercharges $\mathbb{Q}_1$ and $\mathbb{Q}_2$ resp. $\mathbb{Q}$ and $\mathbb{Q}^\dagger$. We will mostly deal with $N = 2$ extended SUSY (and equivalent algebras) in one spatial dimension in this treatise.

2 Supersymmetric Formalism

In this chapter, the idea of SUSY in quantum mechanics will be explained. For a brief derivation in the context of quantum field theories one may consider the referenced literature (e.g. [3]). Here we will concentrate on the quantum mechanical properties of supersymmetric quantum systems.
Supersymmetric quantum mechanics refers to quantum mechanics based on the (stationary) Schrödinger equation

$$\begin{aligned} \mathbb{H}\,\psi &= E\,\psi \quad , \\ \left(\frac{\mathbb{P}^2}{2m} + V(\mathbb{X})\right)\psi &= E\,\psi \quad , \end{aligned} \tag{2.1}$$

taking into account the SUSY algebra presented in (1.3).

2.1 Hamiltonian Factorization and the Superpotential

For the purpose of understanding the mechanism behind SUSY, we will first investigate unbroken cases of SUSY. Let us consider a one-dimensional quantum system described by a Hamiltonian $\mathbb{H}^{(1)}$ with a given, time-independent potential $V^{(1)}(x)$

$$\mathbb{H}^{(1)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(1)}(x) \quad . \tag{2.2}$$

The potential $V^{(1)}(x)$ can be shifted by a constant such that the ground state energy of the Hamiltonian $E_0^{(1)} = 0$. This does not affect the wave functions and scattering properties of the potential at all. It only results in a shifted energy spectrum (by the constant). So, without loss of generality for the following derivations, the Hamiltonian acting on the ground state wave function yields

$$\begin{aligned} \mathbb{H}^{(1)}\,\psi_0(x) &= 0 \quad , \\ -\frac{\hbar^2}{2m}\psi_0''(x) + V^{(1)}(x)\,\psi_0(x) &= 0 \quad , \end{aligned} \tag{2.3}$$

where ψ_0 denotes $\psi_0^{(1)}$ from here on. Relation (2.3) can be rearranged to

$$V^{(1)}(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)} \quad . \tag{2.4}$$

Due to the ground state being nodeless, this expression is well-defined. Thus, the potential is also reversely determined by the ground state wave function. This means that the Hamiltonian can be constructed by either knowing the potential $V^{(1)}(x)$ directly or with the aid of the ground state wave function ψ_0 of the quantum system.
Now we arrive at an important point: The Hamiltonian can be factorized as

$$\mathbb{H}^{(1)} = \mathbb{A}^\dagger\mathbb{A} \tag{2.5}$$

with the operators

$$\mathbb{A} \doteq \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \qquad , \qquad \mathbb{A}^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \quad . \tag{2.6}$$

$W(x)$ is known as the superpotential and its dimension is the square root of energy. Subsequently, $V^{(1)}(x)$ can be written in terms of $W(x)$ (considering the case of unbroken SUSY, we will come back to this later):

$$\begin{aligned}
\mathbb{H}^{(1)} &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(1)}(x) \\
&= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) = \mathbb{A}^\dagger\mathbb{A} \\
\Longrightarrow \quad V^{(1)}(x) &= W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) \quad . \qquad (2.7)
\end{aligned}$$

The latter is referred to as the Riccati equation. Considering the ground state energy condition (2.3) with the factorized Hamiltonian from (2.5) gives

$$-\frac{\hbar^2}{2m}\,\psi_0''(x) + \left(W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x)\right)\psi_0(x) = 0 \quad .$$

Rearranging and using the chain rule for differentiation yields

$$\begin{aligned}
\frac{\psi_0''(x)}{\psi_0(x)} &= \left(\frac{\sqrt{2m}}{\hbar}\,W(x)\right)^2 - \frac{\sqrt{2m}}{\hbar}\,W'(x) \quad , \\
\Longrightarrow \quad \left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2 + \left(\frac{\psi_0'(x)}{\psi_0(x)}\right)' &= \left(\frac{\sqrt{2m}}{\hbar}\,W(x)\right)^2 - \frac{\sqrt{2m}}{\hbar}\,W'(x) \quad .
\end{aligned}$$

One solution of this equation and therefore our superpotential is

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)} = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\,\ln\,\psi_0(x) \quad . \qquad (2.8)$$

As a check, one can plug this relation back into the Riccati equation (2.7) and finally retrieve (2.4) again.

2.2 Partner Potentials

We found that the Hamiltonian can be factorized as in (2.5) – now let us exchange the order of $\mathbb{A}$ and $\mathbb{A}^\dagger$

$$\begin{aligned}
\mathbb{H}^{(2)} &= \mathbb{A}\,\mathbb{A}^\dagger \\
&= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \qquad (2.9) \\
&= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(2)}(x) \quad .
\end{aligned}$$

Hence we have found another Hamiltonian with a new potential

$$V^{(2)}(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \quad . \qquad (2.10)$$

The potentials $V^{(1)}(x)$ and $V^{(2)}(x)$ can be related to the same superpotential $W(x)$, they only differ by

$$V^{(2)}(x) - V^{(1)}(x) = \mathbb{H}^{(2)} - \mathbb{H}^{(1)} = [\mathbb{A},\ \mathbb{A}^\dagger] = 2\,\frac{\hbar}{\sqrt{2m}}\,W'(x) \quad . \qquad (2.11)$$

The two potentials are called supersymmetric partner potentials. In the next subsection we will see that the two quantum systems described by $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ are closely related to each other.
To put the procedure in a nutshell, one starts from a general Hamiltonian $\mathbb{H}$, subtracts the ground state energy $\mathbb{H}^{(1)} \doteq \mathbb{H} - E_0$, determines the superpotential $W(x)$ via (2.8) and finally retrieves the partner potential $V^{(2)}(x)$ resp. the Hamiltonian $\mathbb{H}^{(2)}$ with (2.10). Therefore, the SUSY mechanism can take every potential and produce a likewise solved partner potential. This can and has been used to enlarge the known classes of exactly solvable potentials, one of the main achievements of supersymmetric quantum mechanics.

2.3 Energy Level Degeneracies

What is so special about the two potentials being related by a superpotential? As already mentioned in chapter 1, symmetries lead to degeneracies. As we will see in the following, the two quantum systems related by SUSY basically have the same energy spectra. Thus, every energy level of the supersymmetric system (which contains both Hamiltonians) has two energy eigenstates – every energy level except for the ground state of $\mathbb{H}^{(1)}$. We will come back to the supersymmetric system later: let us first investigate the (positive semi-definite) energy eigenvalues $E_n^{(1,2)} \geq 0$ of the two Hamiltonians. Given the Schrödinger equation for the first quantum system

$$\mathbb{H}^{(1)}\,\psi_n^{(1)} = \mathbb{A}^\dagger \mathbb{A}\,\psi_n^{(1)} = E_n^{(1)}\,\psi_n^{(1)} \quad , \tag{2.12}$$

one obtains for the second quantum system

$$\mathbb{H}^{(2)}\left(\mathbb{A}\,\psi_n^{(1)}\right) = \mathbb{A}\mathbb{A}^\dagger \mathbb{A}\,\psi_n^{(1)} = \mathbb{A}\left(\mathbb{H}^{(1)}\,\psi_n^{(1)}\right) = E_n^{(1)}\left(\mathbb{A}\,\psi_n^{(1)}\right) \quad . \tag{2.13}$$

This means that we found eigenfunctions of $\mathbb{H}^{(2)}$, namely $\mathbb{A}\,\psi_n^{(1)}$, with the eigenvalues $E_n^{(1)}$. Similarly, given the Schrödinger equation for the second quantum system

$$\mathbb{H}^{(2)}\,\psi_n^{(2)} = \mathbb{A}\mathbb{A}^\dagger\,\psi_n^{(2)} = E_n^{(2)}\,\psi_n^{(2)} \quad , \tag{2.14}$$

one retrieves a similar relation for the first quantum system

$$\mathbb{H}^{(1)}\left(\mathbb{A}^\dagger\,\psi_n^{(2)}\right) = \mathbb{A}^\dagger \mathbb{A}\mathbb{A}^\dagger\,\psi_n^{(2)} = \mathbb{A}^\dagger\left(\mathbb{H}^{(2)}\,\psi_n^{(2)}\right) = E_n^{(2)}\left(\mathbb{A}^\dagger\,\psi_n^{(2)}\right) \quad . \tag{2.15}$$

Likewise, we found the eigenfunctions of $\mathbb{H}^{(1)}$ to be $\mathbb{A}^\dagger\,\psi_n^{(2)}$, with corresponding eigenvalues $E_n^{(2)}$.
In summary, the spectra of $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ are equal (Dunne and Feinberg [4] call them "isospectral") except for the ground state which only exists in the first quantum system (which will be further discussed later on). Quantitatively, from relations (2.12) to (2.15) we can see that the two Hamiltonians are related by

$$\begin{aligned} E_n^{(2)} &= E_{n+1}^{(1)} \quad , \quad E_0^{(1)} = 0 \\ \psi_n^{(2)} &= \frac{1}{\sqrt{E_{n+1}^{(1)}}}\,\mathbb{A}\,\psi_{n+1}^{(1)} \quad , \quad \psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}}\,\mathbb{A}^\dagger\,\psi_n^{(2)} \\ &(n = 0, 1, 2, ...) \end{aligned} \tag{2.16}$$

Obviously, the operator $\mathbb{A}$ maps the eigenfunctions of the first to the second quantum system and vice versa, the operator $\mathbb{A}^\dagger$ maps the second Hamiltonian's wave functions to those of the first one. Hereby, $\mathbb{A}$ destroys one node of the eigenfunction and $\mathbb{A}^\dagger$ creates one, whereas $\mathbb{A}$ annihilates the ground state $\psi_0^{(1)}$. Thus, starting from the first system we can construct the whole energy spectrum of the second quantum system including the corresponding wave functions, and reversely the first system can be retrieved from the second Hamiltonian losing only the ground state. This situation is illustrated in figure 2.1.

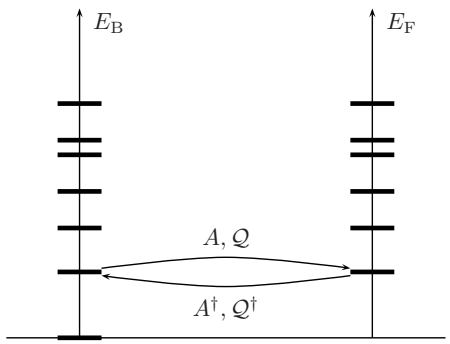

Figure 2.1: Energy levels of two quantum systems related by Supersymmetry. In this unbroken case, every energy level has a supersymmetric partner except for the ground state of the first Hamiltonian – their relation through the SUSY operators $\mathbb{A}$ and $\mathbb{A}^\dagger$ resp. $\mathbb{Q}$ and $\mathbb{Q}^\dagger$ is indicated by arrows. Picture taken from [1]

2.4 Supersymmetric Hamiltonian

In order to better understand the degeneracy of energy levels, we introduce a SUSY Hamiltonian $\mathbb{H}^{(S)}$ considering both quantum systems in one supersymmetric system. $\mathbb{H}^{(S)}$ can be constructed as a 2×2 matrix

$$\mathbb{H}^{(S)} = \begin{pmatrix} \mathbb{H}^{(1)} & 0 \\ 0 & \mathbb{H}^{(2)} \end{pmatrix} = \begin{pmatrix} \mathbb{A}^\dagger \mathbb{A} & 0 \\ 0 & \mathbb{A}\,\mathbb{A}^\dagger \end{pmatrix} \quad . \tag{2.17}$$

Following the symmetry and algebra discussion in the motivation chapter, the supersymmetric Hamiltonian needs to comply with

$$[\mathbb{H}^{(S)},\ \mathbb{Q}_1] = [\mathbb{H}^{(S)},\ \mathbb{Q}_2] = 0 \quad , \tag{2.18}$$

where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are two Hermitian supercharges. Squaring them must result in $\frac{1}{2}\,\mathbb{H}^{(S)}$. So, considering

$$\mathbb{Q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbb{A}^\dagger \\ \mathbb{A} & 0 \end{pmatrix} \quad , \quad \mathbb{Q}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i\,\mathbb{A}^\dagger \\ -\,i\,\mathbb{A} & 0 \end{pmatrix} \quad , \tag{2.19}$$

we find all the requirements for the $N = 2$ extended SUSY algebra (1.3) with $i, k = 1, 2$ fulfilled.

Another way to write the symmetry operators leaving the energy of the system invariant as in (2.18) is

$$\mathbb{H}^{(S)} = \{\mathbb{Q},\ \mathbb{Q}^\dagger\} \tag{2.20}$$

(thus also having factorized the supersymmetric Hamiltonian) with the non-Hermitian supercharge $\mathbb{Q}$ and its Hermitian conjugate $\mathbb{Q}^\dagger$ being chosen as

$$\mathbb{Q} = \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \quad , \quad \mathbb{Q}^\dagger = \begin{pmatrix} 0 & \mathbb{A}^\dagger \\ 0 & 0 \end{pmatrix} \quad . \tag{2.21}$$

These operators are nilpotent due to $\{\mathbb{Q},\ \mathbb{Q}\} = \{\mathbb{Q}^\dagger,\ \mathbb{Q}^\dagger\} = 0$. By the way, this can be traced back to the nilpotence of fermionic creation and annihilation operators. Considering a quantum system with both bosonic and fermionic states, the supercharges can be constructed with their corresponding operators (using the second quantization formalism $|n_{bosons}, n_{fermions}\rangle$ with creation and annihilation operators). In this context, the first Hamiltonian $\mathbb{H}^{(1)}$ is often referred to as the bosonic part of the supersymmetric system with the fermion number being zero $\big(|n_{bosons}, 0\rangle\big)$, whereas the fermionic part $\mathbb{H}^{(2)}$ has a fermion number being one $\big(|n_{bosons}, 1\rangle\big)$: the Hilbert space $\mathcal{H}$ is split into the two classes of either bosonic or fermionic state. For further information one may refer to [3] – but as we know, in SUSY quantum mechanics every potential has a partner potential and thus, here the boson and fermion discussion is not really needed. Switching back to our topic, the latter algebra 2.20 is known as the closed superalgebra $sl(1/1)$. It is related to the afore-mentioned $N = 2$ extended SUSY algebra by

$$\mathbb{Q} = \frac{\mathbb{Q}_1 + i\,\mathbb{Q}_2}{\sqrt{2}} \quad , \quad \mathbb{Q}^\dagger = \frac{\mathbb{Q}_1 - i\,\mathbb{Q}_2}{\sqrt{2}} \quad . \tag{2.22}$$

2.5 Ground State Wave Function

A ground state wave function $\psi_0(x)$ with a physical meaning has the following properties:

$$\hookrightarrow \text{finite absolute integral:} \quad \int dx\ \big|\psi_0(x)\big|^2 = \int dx\ \big(\psi_0^*\,\psi_0\big)(x) < \infty \tag{2.23}$$

(which is referred to as being normalizable)

$$\hookrightarrow \text{therefore zero at infinity:} \quad \psi_0(x \to \pm\infty) = 0 \tag{2.24}$$

$$\hookrightarrow \text{nodeless:} \quad \psi_0(x) \neq 0 \tag{2.25}$$

Additionally, from (2.3) and the linearity of $\mathbb{A}$, $\mathbb{A}^\dagger$ it can be seen that

$$\mathbb{A}\,\psi_0 \equiv \mathbb{A}\,\psi_0^{(1)} = 0 \tag{2.26}$$

In fact, this is equivalent to shifting the first potential such that the ground state has a zero eigenvalue. Equation (2.26) implies that we cannot extract any information about a possible state in the second quantum system which is located at $E_0^{(1)} = 0$: $\psi_0^{(1)}$ has no supersymmetric partner. From the point of view of the SUSY transformations, there exists only a valid state at zero energy in the first quantum system (for exact SUSY). In the following, we will always

choose the first Hamiltonian to have the valid ground state at zero (if at all): Thus, the first quantum system has fermion number zero (as mentioned in the previous subsection).
Along with $V^{(1)}(x)$ (equation (2.7)) and $V^{(2)}(x)$ (equation (2.10)), also the ground state wave function $\psi_0(x)$ can be related to the superpotential $W(x)$

$$\begin{aligned} & \mathbb{A}\,\psi_0(x) \equiv \frac{\hbar}{\sqrt{2m}}\,\frac{d}{dx}\,\psi_0(x) + W(x)\,\psi_0(x) \overset{(2.26)}{=} 0 \\ \Longrightarrow \quad & \frac{d\psi_0}{\psi_0} = -\frac{\sqrt{2m}}{\hbar}\,W(x)\,dx \\ \Longrightarrow \quad & \psi_0(x) = \exp\left(-\frac{\sqrt{2m}}{\hbar}\int_{-\infty}^{x} dy\; W(y)\right) \quad . \end{aligned} \tag{2.27}$$

One could as well consider $\mathbb{A}^\dagger\,\psi_0^{(2)} = 0$ resulting in

$$\psi_0^{(2)}(x) = \exp\left(+\frac{\sqrt{2m}}{\hbar}\int_{-\infty}^{x} dy\; W(y)\right) \quad , \tag{2.28}$$

but this would just exchange the meaning of $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ (and we chose the above way of $\mathbb{H}^{(1)}$ having the zero energy ground state).
Hence, following our conventions, for unbroken SUSY the superpotential $W(x)$ has to be negative for $x \to -\infty$ and positive for $x \to +\infty$ (otherwise the wave function (2.27) would not be normalizable).
The ground state wave function $|0\rangle$ of $\mathbb{H}^{(S)}$ can be defined as the two-dimensional vector

$$|0\rangle = \begin{pmatrix} \psi_0^{(1)}(x) \\ \psi_0^{(2)}(x) \end{pmatrix} \quad . \tag{2.29}$$

Again, supersymmetry is exact resp. unbroken when there exists a supersymmetric state with $\mathbb{Q}_{1\text{ or }2}\;|0\rangle = 0$, which means that its energy eigenvalue is $\mathbb{H}^{(S)}\;|0\rangle = 2\,\mathbb{Q}_i^2\;|0\rangle = 0$. Therefore, only one of the two ground state wave functions can be normalizable and (2.29) can be written as

$$|0\rangle = \begin{pmatrix} \psi_0^{(1)}(x) \\ 0 \end{pmatrix} \quad , \tag{2.30}$$

where $\psi_0^{(1)}(x)$ is defined by (2.27).

2.6 Broken Supersymmetry

In the previous discussion we have always been considering unbroken SUSY – now let us investigate the broken case.
In physics, spontaneous symmetry breaking means that a generator $\mathbb{T}$ of a transformation (corresponding to a symmetry of the Hamiltonian, i.e. $[\mathbb{H},\ \mathbb{T}] = 0$) does not leave the ground state invariant: $\mathbb{T}\,|0\rangle \neq 0$. The ground state does not respect the symmetry.
Earlier, we discussed the two cases of exact SUSY with either $\mathbb{H}^{(1)}$ or $\mathbb{H}^{(2)}$ having a ground state at zero energy. Starting from $\mathbb{A}\,\psi_0^{(1)} = 0$ resulted in $W(x)$ being strictly negative at negative infinite x-values and being strictly positive at positive infinite x-values (see figure 2.2) so that the wave function in eq. (2.27)

is normalizable (and $\mathbb{H}^{(1)}$ has the zero ground state). In contrast, when starting from $\mathbb{A}^\dagger\,\psi_0^{(2)} = 0$ (see figure 2.3), $W(x)$ behaves just the other way round (and $\mathbb{H}^{(2)}$ has the zero-energy ground state). Then one can flip $W(x) \;\rightarrow\; -W(x)$ in order to get back our conventions having the first Hamiltonian as the bosonic part of our supersymmetric system.

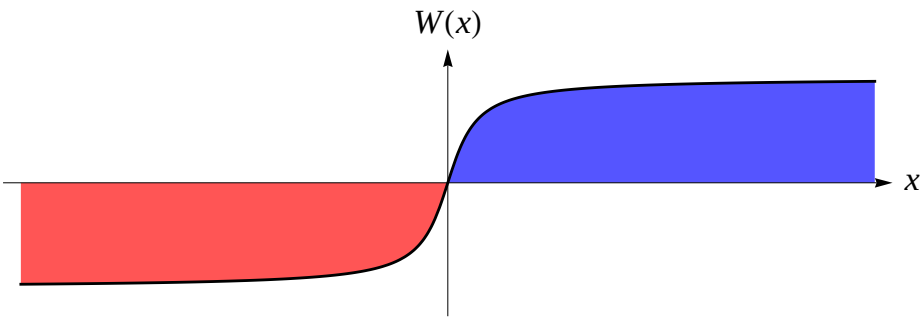

Figure 2.2: Superpotential corresponding to unbroken SUSY with $\mathbb{H}^{(1)}$ having a ground state at $E = 0$.

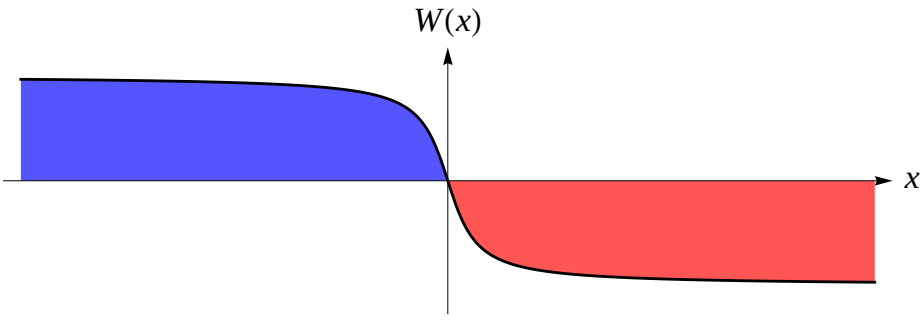

Figure 2.3: Superpotential corresponding to unbroken SUSY with $\mathbb{H}^{(2)}$ having a ground state at $E = 0$.

Again, due to the demand on the ground state wave function to be normalizable, it must vanish at positive and negative infinity because the domain ranges over the whole real axis. To be concrete, this means that for $|x| \rightarrow \infty$ the exponential function of (2.27) $\big(\psi_0(x) \propto \exp(-\ldots)\big)$ or (2.28) $\big(\psi_0(x) \propto \exp(+\ldots)\big)$ has to converge to zero, resp. its argument has to converge to $-\infty$. Thus, $\psi_0^{(1)}$ is a ground state at $E = 0$ when

$$\int_{-\infty}^{0} dy\; W(y) = -\infty \qquad , \qquad \int_{0}^{\infty} dy\; W(y) = +\infty \quad , \tag{2.31}$$

whereas $\psi_0^{(2)}$ is a ground state at $E = 0$ when

$$\int_{-\infty}^{0} dy\; W(y) = +\infty \qquad , \qquad \int_{0}^{\infty} dy\; W(y) = -\infty \quad . \tag{2.32}$$

Obviously, at most only one of the two conditions can be fulfilled. Given this, SUSY is unbroken.
However, what happens if the wave functions in neither (2.27) nor (2.28) are normalizable because of $W(x)$ being even and not fulfilling any of the above conditions (see figure 2.4)? An analytic example is given in section 3.2. This corresponds to the case of broken SUSY where neither $\mathbb{Q}$ nor $\mathbb{Q}^\dagger$ annihilates the ground state $|0\rangle$ defined in (2.29) (as described for general spontaneous symmetry breaking, $\mathbb{Q}$ being the transformation generator). Thus, creating the factorized Hamiltonians, we cannot retrieve a ground state at $E = 0$. Broken SUSY means that there exists no valid state at zero energy in both $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$.

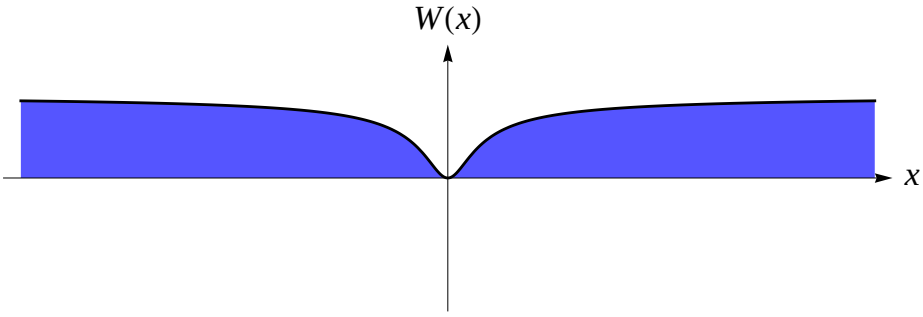

Figure 2.4: Superpotential corresponding to broken SUSY not allowing a ground state at $E = 0$.

We conclude from this that the ground states of both quantum systems have a positive non-zero energy value. All the other SUSY statements and results already discussed, which do not involve the ground state or the counting, are still valid. The spectra of both $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ are therefore exactly equal, even the ground state with $E_0 > 0$ is degenerate in contrast to exact SUSY. The picture we get from this is outlined in figure 2.5.

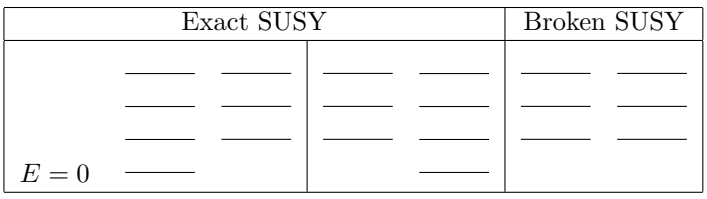

Figure 2.5: Comparison between the degenerate spectra of an unbroken and a broken supersymmetric quantum system. Each of them still has a relation of the respective energy levels through the operators $\mathbb{A}$ and $\mathbb{A}^\dagger$, the only difference is the state counting.

The two operators $\mathbb{A}$ and $\mathbb{A}^\dagger$ now connect equally labeled energy levels and do not change the number of nodes in the wave functions anymore:

$$\begin{aligned} E_n^{(2)} = E_n^{(1)} > 0 \quad &, \quad E_0^{(1)} = 0 \\ \psi_n^{(2)} = \frac{1}{\sqrt{E_n^{(1)}}} \mathbb{A}\, \psi_n^{(1)} \quad &, \quad \psi_n^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} \mathbb{A}^\dagger\, \psi_n^{(2)} \\ (n = 0, 1, 2, ...) \end{aligned} \tag{2.33}$$

In summary, starting from a superpotential $W(x)$, we can easily determine whether SUSY is broken or not: if the sign of $W(x)$ is different when comparing both $x \to +\infty$ and $x \to -\infty$ cases then generally the ground state wave function is normalizable. If the sign is the same and $W(x)$ is even, then SUSY is always broken.
Let us finally introduce a topological quantum number: the Witten index Δ can be used to determine whether SUSY is exact or broken. It is often discussed in the context of quantum field theories. The Witten index is defined as

$$\Delta \doteq n_b^0 - n_f^0 \tag{2.34}$$

where n_b^0 and n_f^0 represent the number of bosonic and fermionic states at $E = 0$ (again, the bosonic state corresponds to $\mathbb{H}^{(1)}$ and the fermionic one to $\mathbb{H}^{(2)}$). For broken SUSY, the Witten index is always zero because all states emerge in supersymmetrically related pairs. However, $\Delta = 1$ reflects exact SUSY because here the ground state does not have a supersymmetric partner. Considering only one dimension $d = 1$, this works in both ways: exact SUSY also means $\Delta = 1$.

2.7 Scattering Properties

Later on, we will study the scattering properties of the delta-well potential.
In [5] the relations between the reflection and transmission coefficients are derived for a general potential and its superpartner. The argument uses that not only the discrete energy spectra of two supersymmetrically related Hamiltonians are equal – in fact the continuum wave functions of $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ also have the same energy. Defining

$$W_- \doteq \lim_{x \to -\infty} W(x) \quad , \quad W_+ \doteq \lim_{x \to \infty} W(x) \quad , \tag{2.35}$$

the general coefficients are related by

$$r_1(k) = \left(\frac{W_- + ik}{W_- - ik} \right) r_2(k) \quad , \quad t_1(k) = \left(\frac{W_+ - ik'}{W_- - ik} \right) t_2(k) \tag{2.36}$$

$$\text{where} \quad k \doteq \sqrt{E - W_-^2} \quad , \quad k' \doteq \sqrt{E - W_+^2} \quad .$$

One can see that the partner potentials have the same reflection and transmission probabilities $|r_1|^2 = |r_2|^2$ and $|t_1|^2 = |t_2|^2$. If one of the two partner Hamiltonians describes a free particle which means that the potential is essentially constant, then the superpartner is reflectionless as well.

3 Applications

3.1 Infinite Square Well Potential

Let us consider a first example following the procedure given in section 2.2: the famous infinite square well potential (cf. [6], [7]) which traditionally is the first candidate to be investigated in quantum mechanics. It is defined by

$$V(x) \doteq \begin{cases} 0, & 0 < x < a \\ \infty, & \text{else} \end{cases} \quad . \tag{3.1}$$

Figure 3.1: The infinite square well potential with a length of a .

A particle of mass m in this potential can move between 0 and a but obviously not outside, thus its wave function underlies the restriction

$$\psi(0) = \psi(a) = 0 \quad . \tag{3.2}$$

Possible eigenfunctions of the particle's Hamiltonian

$$\mathbb{H}\,\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) \tag{3.3}$$

are therefore the sine functions, the ansatz $\psi(x) = c\,\sin(d\,x)$ successfully solves the Schrödinger equation for the infinite square well.
The normalized ground state wave function of the particle is then easily found to be

$$\psi_0(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{\pi\,x}{a}\right) \tag{3.4}$$

along with its energy eigenvalue

$$E_0 = \frac{\pi^2\,\hbar^2}{2m\,a^2} \quad . \tag{3.5}$$

Now, following the mentioned procedure, one firstly subtracts the ground state energy E_0 from the potential: $V^{(1)}(x) \doteq V(x) - E_0$. Using equation (2.8), the superpotential $W(x)$ is found to be

$$W(x) = -\sqrt{E_0}\,\left(\tan\left(\frac{\pi\,x}{a}\right)\right)^{-1} = -\frac{\pi\,\hbar}{\sqrt{2m}\,a}\,\frac{1}{\tan\left(\frac{\pi\,x}{a}\right)} \quad , \tag{3.6}$$

and a check with $V^{(1)}(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x)$ (2.7) returns the shifted potential again.
Once the superpotential has been determined, we are ready to find the SUSY partner potential of the infinite square well potential with the aid of (2.10)

$$\begin{aligned} V^{(2)} &= E_0 \, \frac{1}{\tan^2\left(\frac{\pi\, x}{a}\right)} - \frac{\hbar}{\sqrt{2m}} \sqrt{E_0} \, \frac{\pi}{a} \left(\frac{-\sin^2\left(\frac{\pi\, x}{a}\right) - \cos^2\left(\frac{\pi\, x}{a}\right)}{\sin^2\left(\frac{\pi\, x}{a}\right)} \right) \\ &= E_0 \left(\frac{1}{\tan^2\left(\frac{\pi\, x}{a}\right)} + \frac{1}{\sin^2\left(\frac{\pi\, x}{a}\right)} \right) \\ &= \frac{\pi^2\, \hbar^2}{2m\, a^2} \, \frac{\cos^2\left(\frac{\pi\, x}{a}\right) + 1}{\sin^2\left(\frac{\pi\, x}{a}\right)} \quad . \end{aligned} \tag{3.7}$$

This potential goes naturally to infinity at $x = 0$ and $x = a$ (due to the sine in the denominator) and outside of these boundaries it is still defined as infinity anyway (cf. the definition of the infinite square well, there we have zero probability of the particle's presence). The nature of SUSY now tells us that both the square well potential $V^{(1)}$ and its superpartner $V^{(2)}$ have the same energy levels, only the ground state exclusively exists for $\mathbb{H}^{(1)}$ because the lowest value of the partner potential at $\frac{a}{2}$ is bigger than zero. Thus $\mathbb{H}^{(2)}$ cannot have a bound state at $E = 0$. The two partner potentials and the superpotential are depicted in figure 3.2.

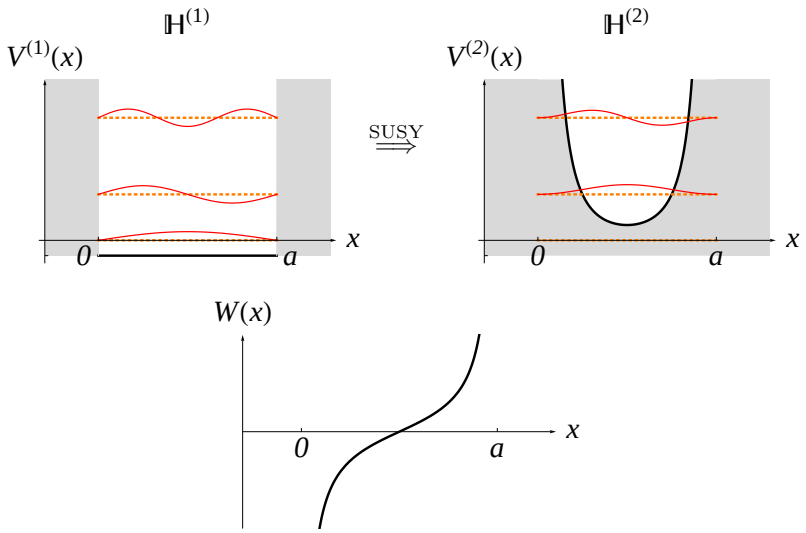

Figure 3.2: Top left: infinite square well potential with the first three energy levels and the corresponding eigenfunctions; top right: partner potential with the same spectrum and the first two eigenfunctions; bottom: super potential $W(x)$ which connects the two potentials.

In conclusion, the infinite square well potential has a superpartner with an unbroken SUSY relation. All the energy levels which we find determined for $\mathbb{H}^{(1)}$ as $E_n = \frac{n^2\, \pi^2\, \hbar^2}{2m\, a^2} - E_0$ are the same for $\mathbb{H}^{(2)}$ from $n = 2$ on – and all the corresponding eigenfunctions from $\mathbb{H}^{(2)}$ are retrieved via $\mathbb{A}\, \psi_{n+1}$.

3.2 Polynomial Superpotentials

A very nice example for breaking SUSY can be studied using the polynomial superpotentials. They are defined by

$$W(x) \doteq c\,x^n \qquad n \in \mathbb{N} \quad . \tag{3.8}$$

We devide them into an even $n = 2m$ and an odd $n = 2m - 1$ class ($m \in \mathbb{N}$). The odd polynomials give rise to an exact SUSY quantum system since the limiting values of $W(|x| \to \infty)$ have different signs. The two partner potentials are even:

$$\begin{aligned} V^{(1)}(x) = W^2(x) - W'(x) &= c^2\,x^{2n} - n\,c\,x^{n-1} \\ &= c^2\,x^{4m-2} - (2m-1)\,c\,x^{2m-2} \quad , \\ V^{(2)}(x) = W^2(x) + W'(x) &= c^2\,x^{2n} + n\,c\,x^{n-1} \\ &= c^2\,x^{4m-2} + (2m-1)\,c\,x^{2m-2} \quad . \end{aligned}$$

If $c > 0$ then $V^{(1)}(x) < V^{(2)}(x)$ and we have the single ground state at $\mathbb{H}^{(1)}$; if $c < 0$ we can take $-W(x)$ to achieve our conventions. For $n = 1$, we get the harmonic oscillator which is discussed in section 3.3. All higher values $n = 3, 5, 7, ...$ yield a single well potential $V^{(2)}(x)$ with its minimum at zero and a double well potential $V^{(1)}(x)$ with a local maximum at zero and its local minima symmetrically located to both sides (see figure 3.3). In principle, we should find the ground state of the first Hamiltonian to sit in these sinks.
Now, considering the even polynomials $n = 2m$ we obtain broken SUSY – why? The limiting values of the superpotential at infinity are both either positive or negative. The derivative of $W(x)$ is not even and therefore our two partner potentials are just related by a reflection at $x = 0$ (see figure 3.3). Therefore, their spectra naturally need to be exactly equal. Thus, we obtain broken SUSY.

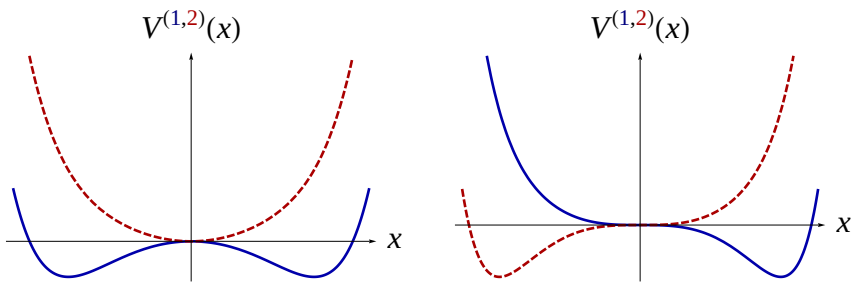

Figure 3.3: Left: odd superpotential ⇒ exact SUSY;
right: even superpotential ⇒ broken SUSY

3.3 Harmonic Oscillator

The harmonic oscillator (cf. [6]) can be derived from the first odd polynomial superpotential described in the last section. It is another standard quantum mechanical problem studied quite often. Here, one can easily see the powerful feature of SUSY quantum mechanics to retrieve whole spectra of potentials by investigating their partner potentials, the partners of those and the whole following series of Hamiltonians each related by SUSY.
The potential of the quantum harmonic oscillator looks like

$$V(x) \doteq \frac{1}{2}\, m\, \omega^2\, x^2 \quad . \tag{3.9}$$

As already said, the corresponding superpotential is a linear polynomial

$$W(x) = c \cdot x = \sqrt{\frac{m}{2}}\, \omega\, x \quad . \tag{3.10}$$

Due to the constant derivative of $W(x)$, we find the partner potentials to be just shifted by a constant:

$$\begin{aligned} V^{(1)}(x) &= \frac{1}{2} m\omega^2 x^2 - \frac{\hbar\omega}{2} \quad , \\ V^{(2)}(x) &= \frac{1}{2} m\omega^2 x^2 + \frac{\hbar\omega}{2} \quad , \\ V^{(2)}(x) - V^{(1)}(x) &= \hbar\omega \quad . \end{aligned} \tag{3.11}$$

So, due to the subtraction of $E_0 = \frac{\hbar\omega}{2}$, we have the ground state of $\mathbb{H}^{(1)}$ at $E = 0$. The second Hamiltonian describes the same physical system, so its ground state compared to $\mathbb{H}^{(1)}$ is shifted by $\Delta E = \hbar\omega$ – which is the supersymmetric partner of the first excited state of $\mathbb{H}^{(1)}$. Hence, we now know the first two energy levels of the harmonic oscillator $\mathbb{H}^{(1)}$ to be located at $E_0^{(1)} = 0$ and $E_1^{(1)} = \hbar\omega$. Playing the same game for the shifted harmonic oscillator $\mathbb{H}^{(2)}$, we can retrieve a third Hamiltonian as a partner potential to the second one: of course again, this is just a harmonic oscillator shifted by the same constant ΔE. Thus, its ground state is located at $E_0^{(3)} = 2\hbar\omega$ which is a supersymmetric partner to the first excitated state of $\mathbb{H}^{(2)}$– and the latter is supersymmetrically related back to the first Hamiltonian: We found the second excited state of $\mathbb{H}^{(1)}$ to be $E_2^{(1)} = E_1^{(2)} = 2\hbar\omega$. One can continue with this on to a fourth, fifth and nth Hamiltonian, by doing this we obtain the entire spectrum of the harmonic oscillator:

$$E_n^{(1)} = \sum_{i=1}^{n} \Delta E = n\hbar\omega \quad . \tag{3.12}$$

In conclusion, just by knowing the ground state of the harmonic oscillator and finding its superpartner to be the same system shifted by a constant, we retrieved the whole energy spectrum with the aid of the SUSY related Hamiltonian series. Therefore, one just needs to know the ground state of the respective superpartner along with its ground state eigenfunction in order to determine the next superpartner – this was very easy for the harmonic oscillator because its physical behaviour including the ground state did not change at all.

Similar to the energy eigenstates, one can also find all the eigenfunctions by knowing the ground state wave function

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \quad . \tag{3.13}$$

Due to the fact that $\psi_0(x)$ stays the same for all Hamiltonians in the SUSY partner series, we can determine the higher eigenfunctions via

$$\psi_n(x) = \underbrace{\mathbb{A}^\dagger \mathbb{A}^\dagger \cdots \mathbb{A}^\dagger}_{n} \, \psi_0(x) \quad . \tag{3.14}$$

For a general potential, the respective ground state wavefunctions would naturally need to be determined at each step. The Hamiltonian series and their corresponding energy levels are depicted in figure (3.4).

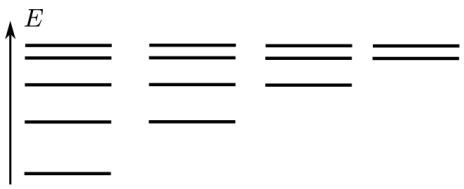

Figure 3.4: A series of Hamiltonians connected by SUSY operators $\mathbb{A}$ and $\mathbb{A}^\dagger$. With each step in the SUSY partner series, one loses the corresponding ground state. The picture was taken from *http://www.sunclipse.org/?p=466*

Just as a side remark, the harmonic oscillator's spectrum is usually determined by using the ladder operators $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{a} \doteq \sqrt{\frac{m\omega}{2\hbar}} \left(\mathbb{X} + \frac{i}{m\omega}\,\mathbb{P}\right) \quad , \tag{3.15}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\mathbb{X} - \frac{i}{m\omega}\,\mathbb{P}\right) \quad . \tag{3.16}$$

In position space, we find them to be directly related to the SUSY operators $\mathbb{A}$ and $\mathbb{A}^\dagger$ by just using different units: $\mathbb{A} = \sqrt{\hbar\omega}\,\hat{a}$. This gives another intuitive explanation for the behaviour of the SUSY operators as described in section 2.3. We already mentioned at the beginning of this subsection that the mechanism to use the SUSY Hamiltonian series in order to work out a potential's spectrum is a great achievement of SUSY quantum mechanics. The harmonic oscillator belongs to a special class known as shape-invariant potentials (SIP). However, it is a very simple example because of the never changing difference ΔE between the ground states. Generally, the constant may change over the series (but still needs to be constant with respect to x which is not given by definition: $\Delta E = 2W'(x) \overset{!}{=} const$). One then sums up these constants as in (3.12) to get the corresponding eigenvalue of the first Hamiltonian. For a deeper insight, please refer to [8] where shape-invariant potentials are examined in detail. This topic is also treated very intensively in [5].

3.4 Delta-Well Potential

The delta-well potential (cf. [6]) models a contact interaction with a certain strength. The Dirac delta function vanishes everywhere except for one point ($x = 0$) where its value goes to infinity. Nevertheless its integral has a constant value which defines its strength. We will consider an attractive delta-well potential (sketched in figure 3.5):

$$
\begin{aligned}
V(x) &\doteq -D \cdot \delta(x) = \begin{cases} -\infty, & x = 0 \\ 0, & else \end{cases} \quad , \qquad D > 0 \quad , \\
\int_{-\infty}^{\infty} dx\; V(x) &= -D \cdot \underbrace{\int_{-\infty}^{\infty} dx\; \delta(x)}_{\doteq 1} = -D \quad .
\end{aligned}
\tag{3.17}
$$

The Dirac delta function can, for instance, be obtained by squeezing a Gaussian to an infinitely small width while keeping the total area constant.

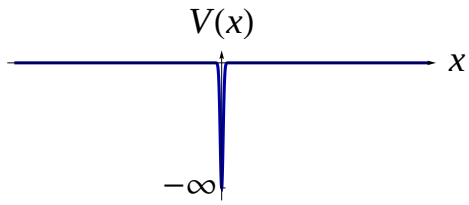

Figure 3.5: Sketch of an attractive delta-well potential which vanishes everywhere except for $x = 0$ where it goes to $-\infty$.

The attractive delta-well potential has one bound state at

$$
E_0 = -\frac{mD^2}{2\hbar^2} \quad , \tag{3.18}
$$

with a ground state wave function

$$
\psi_0(x) = \frac{\sqrt{mD}}{\hbar}\; \exp\left(-\frac{mD}{\hbar^2}\,|x|\right) \quad . \tag{3.19}
$$

Analyzing the demanded properties of the superpotential $W(x)$, it needs to jump discontinuously upwards at $x = 0$ and the squared value of $W(x)$ needs to return E_0. Using (2.8), we find

$$
W(x) = \begin{cases} -\sqrt{\frac{m}{2}}\,\frac{D}{\hbar}, & x < 0 \\ +\sqrt{\frac{m}{2}}\,\frac{D}{\hbar}, & x > 0 \end{cases} \tag{3.20}
$$

The negative derivative of $W(x)$ gives the attractive delta peak at $x = 0$. In (2.10) the sign of the superpotential's derivative changes to be positive and thus we get a flipped delta peak as a barrier

$$
\begin{aligned}
V^{(1)}(x) &= -D\,\delta(x) + \frac{mD^2}{2\hbar^2} \quad , \\
V^{(2)}(x) &= +D\,\delta(x) + \frac{mD^2}{2\hbar^2} \quad .
\end{aligned}
\tag{3.21}
$$

The positive delta-barrier potential naturally has no bound state. In this SUSY system, the loss of the ground state is demonstrated very nicely, see figure 3.6.

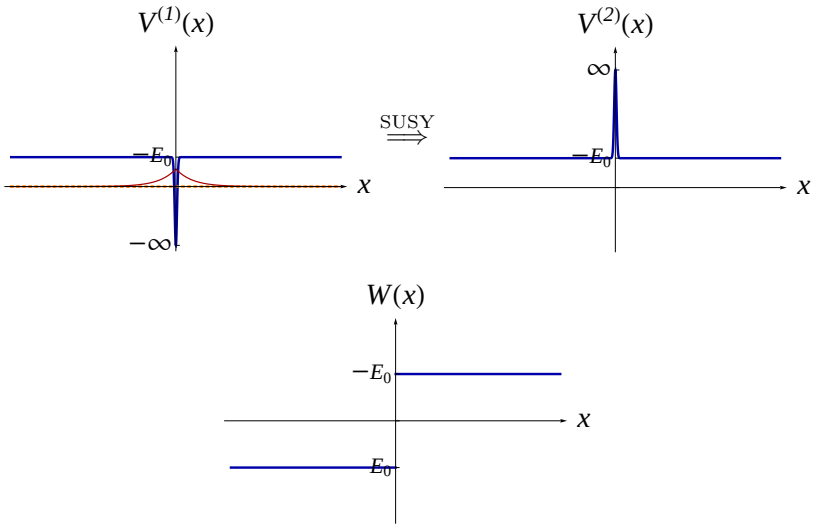

Figure 3.6: Top left: attractive delta-well potential with one bound state; top right: delta-barrier potential without bound states; bottom: superpotential with discontinuous upward jump generating the Dirac delta function.

Let us now investigate the scattering properties of the delta-well and barrier potential. We consider plain waves traversing real space, which are scattered at the delta peak located at $x = 0$:

$$\psi(x) = \begin{cases} A_{to\ right}\, e^{ikx} + A_{to\ left}\, e^{-ikx}, & x < 0 \\ B_{to\ right}\, e^{ikx} + B_{to\ left}\, e^{-ikx}, & x > 0 \end{cases} \quad . \tag{3.22}$$

The situation can be simplified such that we only have to examine waves coming from negative infinity. For the coefficients this means: $A_{to\ right} = 1$ and $B_{to\ left} = 0$. The amplitude $A_{to\ left}$ then resembles the reflection coefficient r and $B_{to\ right}$ to the transmission coefficient t. The wave function now takes the form

$$\psi(x) = \begin{cases} e^{ikx} + r\, e^{-ikx}, & x < 0 \\ t\, e^{ikx}, & x > 0 \end{cases} \quad . \tag{3.23}$$

The wave function and its derivative need to be continuous at $x = 0$ which imposes conditions on its coefficients. On the one hand, the first requirement yields

$$\psi(x \nearrow 0) \stackrel{!}{=} \psi(x \searrow 0) \quad \Longrightarrow \quad 1 + r = t \quad . \tag{3.24}$$

On the other hand, we closely examine the time-independent Schrödinger equation $\mathbb{H}^{(1)}\,\psi(x) = E\,\psi(x)$ resp. $\mathbb{H}^{(2)}\,\psi(x) = E\,\psi(x)$ around the origin at an interval $[-\epsilon, \epsilon]$:

$$\begin{aligned}
& \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \mp D\,\delta(x) - E_0 \right] \psi(x) = E\,\psi(x) \\
\overset{\int dx}{\Longrightarrow} \quad & -\frac{\hbar^2}{2m}\int\limits_{-\epsilon}^{\epsilon} dx\;\psi''(x) \mp D\int\limits_{-\epsilon}^{\epsilon} dx\;\delta(x)\,\psi(x) \\
& \qquad + \frac{mD^2}{2\hbar^2}\underbrace{\int\limits_{-\epsilon}^{\epsilon} dx\;\psi(x)}_{\lim\limits_{\epsilon\to 0}=0} = E\underbrace{\int\limits_{-\epsilon}^{\epsilon} dx\;\psi(x)}_{\lim\limits_{\epsilon\to 0}=0} \\
\overset{\epsilon\to 0}{\Longrightarrow} \quad & \frac{\hbar^2}{2m}\Big(\psi'(x\searrow 0) - \psi'(x\nearrow 0)\Big) \pm D\,\psi(0) = 0 \\
\Longrightarrow \quad & ik\Big(\underbrace{(t) - (1-r)}_{\overset{(3.24)}{=}2r}\Big) = \mp\frac{2mD}{\hbar^2}\,(1+r) \\
\Longrightarrow \quad & r_{1,2}(k) = \frac{1}{\mp\frac{ik\hbar^2}{mD} - 1} \qquad , \qquad t_{1,2}(k) = \frac{1}{\pm\frac{mD}{ik\hbar^2} + 1} \quad . \qquad (3.25)
\end{aligned}$$

In the $\pm, \mp$ signs, the upper one belongs to the attractive delta-well and the lower one to the repulsive delta-barrier. Thus, we see that the reflection and transmission probabilities are the same for both partner potentials as pointed out in section 2.7 – the signs change only for the terms with an imaginary unit i (due to this the absolute value of r_1, r_2 resp. t_1, t_2 are the same). General scattering coefficients have been given in (2.36): applying these to the delta-well potential, one immediately finds the same results with

$$W_+ = \sqrt{\frac{m}{2}}\,\frac{D}{\hbar} = -W_- \quad .$$

4 Periodic Potentials

In solid state physics, one often has to analyse crystal lattices which give rise to periodic potentials and the behaviour of electrons propagating in them. Up to now, we mostly considered potentials with a discrete spectrum which are defined on the entire real axis and we investigated their SUSY partners. In the following, we will investigate periodic potentials which are defined on a finite interval with a length a and which are repeated infinitely on the real axis. From a particle's point of view, one experiences the same physical properties of the potential at a location shifted by an integer multiple of the lattice vector a. The potential therefore has a discrete symmetry. The translation operator $\mathbb{T}(a)$ commutes with the Hamiltonian and hence a conserved quantity emerges.
Periodic potentials differ from ordinary potentials in so far as they have continuous energy bands instead of discrete energy levels: for instance an electron in an ion lattice. Its wave function as a stationary solution of the Schrödinger equation has an energy value in one of the bands. The electron's afore-mentioned conserved property is known as the Bloch momentum, quasi-momentum or crystal momentum $-\frac{\pi}{a} \leq q \leq \frac{\pi}{a}$, it determines the exact energy within the band. Here, the respectively lower band edge corresponds to a Bloch momentum of $q = 0$ and the upper band edge corresponds to a Bloch momentum of $q = \pm\frac{\pi}{a}$. All the energies within a band are twofold degenerate except for the lower band edge which marks the local minimum of energy inside a band. The band edges of a periodic potential are referred to as the Hamiltonian's eigenvalues along with their eigenfunctions. Therefore, the lower edge of the lowest band embodies the ground state of the periodic potential.
Unfortunately, most periodic potential's spectra cannot be solved analytically due to arising transcendental equations. Even in one dimension, very few analytically solvable examples exist. As we already saw, using SUSY one can find partner potentials with equal spectra. In the case of periodic potentials, one can therefore enlarge the class of analytically solvable problems. We will see that the SUSY properties of periodic potentials slightly change in comparison to ordinary potentials. This is mainly due to the finiteness of the potentials definition interval (which is infinitely repeated). The normalizability of the ground state wave function is no issue anymore, and thus the ground states in eq. 2.27 and 2.28 can exist in both SUSY partners. Let us become more concrete now and analyse the breaking of SUSY in more detail during the next subsection.
In the following, we will always choose units such that $\hbar = 2m \equiv 1$: by doing this we get rid of all units and can concentrate on the essentials. For a comparison between the important relations in both unit systems, please refer to appendix A. This chapter is based upon [5], [9] and [10].

4.1 Breaking of SUSY for Periodic Potentials

Without loss of generality, the solutions $\psi(x)$ of the stationary Schrödinger equation with a periodic potential $V(x + a) = V(x)$ can be written such that they fulfill the Bloch condition

$$\psi(x + a) = e^{iqa}\,\psi(x) \quad , \tag{4.1}$$

where q depicts the crystal momentum again. Thus, we see that the lower band edge ($q = 0$) wave functions and, in particular, the nodeless zero modes $\psi_0(x)$

are strictly periodic and have the same period a as the lattice: $\psi(x+a) = \psi(x)$. This means that the superpotential $W(x)$ related to $\psi_0(x)$ by (2.8) as well as the partner potential $V^{(2)}(x)$ also have the same period a. We can examine everything around SUSY within one period of the periodic potential.
As already mentioned above, the ground state is always normalizable for a finite interval as for instance $[0, a]$. Therefore, the wave functions of both eq. (2.27) and (2.28) belong to the Hilbert space and are valid zero modes for $V^{(1)}$ and $V^{(2)}$. In particular, we find them to be related by

$$\psi_0^{(1,2)}(x) = \exp\left(\pm \int_0^x dy\ W(y)\right) \tag{4.2}$$

$$\Longrightarrow \psi_0^{(2)}(x) = \frac{1}{\psi_0^{(1)}(x)} \qquad \text{(unnormalized)} \quad , \tag{4.3}$$

which is well-defined as both zero modes are nodeless. In order to find out whether they are actually valid wave functions, we investigate their behaviour under a discrete translation a which has to satisfy the Bloch condition (4.1)

$$\begin{aligned}
\psi_0(x+a) &= \exp\left(\pm \int_0^{x+a} dy\ W(y)\right) \\
&= \exp\left(\pm \int_0^{x} dy\ W(y) \pm \int_x^{x+a} dy\ W(y)\right) \\
&= \exp\left(\pm \int_x^{x+a} dy\ W(y)\right) \psi_0(x) \\
&\overset{!}{=} \psi_0(x) \\
\Longrightarrow 2n\pi i &= \int_x^{x+a} dy\ W(y) \quad , \qquad n \in \mathbb{Z} \quad .
\end{aligned}$$

Since the right-hand side of the last equation is real (because $W(x)$ is always real for a real initial potential $V^{(1)}(x)$), we find $\psi_0^{(1,2)}(x)$ as in (4.2) to only satisfy the Bloch condition when

$$\int_x^{x+a} dy\ W(y) = 0 \quad . \tag{4.4}$$

Thus, both partner Hamiltonians $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ have the ground states of eq. (4.2) (which have been derived for $E = 0$) only when (4.4) is satisfied. This means, that (4.4) is a condition determining whether SUSY is exact or broken. So, in contrast to ordinary potentials ($\Delta = 1 \ \Rightarrow$ exact SUSY), periodic potentials have exactly the same spectra for both exact and broken SUSY: the Witten index always gives $\Delta = 0$ in the periodic case. When the Witten index is zero, one needs further investigation whether SUSY is broken or not which is why we have to go back to the periodic breaking condition (4.4). The energy levels from both spectra are related by $E_n^{(1)} = E_n^{(2)}$. The mapping between the eigenfunctions turns out to be directly $\psi_n^{(2)}(x) \propto \mathbb{A}\,\psi_n^{(1)}(x)$ resp. $\psi_n^{(1)}(x) \propto \mathbb{A}^\dagger\,\psi_n^{(2)}(x)$ and only in the exact SUSY case, the two zero modes are not supersymmetrically related by the last two relations (because they are always mapped to zero, $\mathbb{H}\,\psi_0 = 0 = E\,\psi_0$).

For the case of exact SUSY, one can easily show that the superpartner of the second Hamiltonian $\mathbb{H}^{(2)}$ turns out to be the first potential again – i.e. one can only retrieve at most one different superpartner from the initial periodic potential

$$\begin{aligned}
W^{(1)}(x) &= -\frac{d}{dx}\ln\Big(\psi_0^{(1)}(x)\Big) \quad , \\
W^{(2)}(x) &= -\frac{d}{dx}\ln\Big(\psi_0^{(2)}(x)\Big) \\
&\overset{(4.3)}{=} -\frac{d}{dx}\ln\left(\frac{1}{\psi_0^{(1)}(x)}\right) \\
&= \frac{d}{dx}\ln\Big(\psi_0^{(1)}(x)\Big) \\
\Longrightarrow W^{(2)}(x) &= -W^{(1)}(x) \quad . \qquad (4.5)
\end{aligned}$$

Thus, the signs of the superpotential's derivative in (2.7) and (2.10) just change and we get back the first potential

$$\begin{aligned}
V^{(1)}(x) &= \Big(W^{(1)}(x)\Big)^2 - \frac{d}{dx}W^{(1)}(x) \quad , \\
V^{(2)}(x) &= \Big(W^{(1)}(x)\Big)^2 + \frac{d}{dx}W^{(1)}(x) = \Big(W^{(2)}(x)\Big)^2 - \frac{d}{dx}W^{(2)}(x) \quad , \\
V^{(3)}(x) &= \Big(W^{(2)}(x)\Big)^2 + \frac{d}{dx}W^{(2)}(x) = \Big(W^{(1)}(x)\Big)^2 - \frac{d}{dx}W^{(1)}(x) \\
\Longrightarrow V^{(3)}(x) &\equiv V^{(1)}(x) \quad . \qquad (4.6)
\end{aligned}$$

For special cases of superpotentials, (4.4) is trivially satisfied and hence SUSY is unbroken, e.g. superpotentials which are antisymmetric on a half-period

$$W(x+\frac{a}{2}) = -W(x) \quad . \qquad (4.7)$$

Consequently, we obtain for the partner potentials

$$\begin{aligned}
V^{(1,2)}(x+\frac{a}{2}) &\equiv W^2(x+\frac{a}{2}) \mp W'(x+\frac{a}{2}) \\
&\overset{(4.7)}{=} W^2(x) \pm W'(x) \\
\Longrightarrow V^{(1,2)}(x+\frac{a}{2}) &= V^{(2,1)} \quad . \qquad (4.8)
\end{aligned}$$

This means that the two partner potentials are essentially the same potential just translated by half a period. Thus, we get no physical information about a new potential from SUSY – the potential $V^{(1)}(x)$ is said to be "self-isospectral". This term is used for all potentials which reproduce a translated and/or reflected version of themselves under SUSY transformations. The latter applies for superpotentials which are even functions of x (also compare for instance with the ordinary even polynomial superpotentials in section 3.2):

$$W(-x) = W(x) \quad , \qquad (4.9)$$

and which satisfy condition (4.4). The superpotential's derivative is odd (as in the case of the polynomials) and therefore

$$V^{(1,2)}(-x) \equiv V^{(2,1)}(x) \quad , \tag{4.10}$$

which implies that the superpotentials are just reflections of each other and hence again self-isospectral.

4.2 Periodic Attractive Delta-Well Potential

Let us investigate a first periodic potential: the Kronig-Penney model is a classic text book example (cf. [7] or [11]) often used to demonstrate the band structure of periodic potentials. It consists of periodically repeated delta peaks which we already analysed as an ordinary single peak potential in section 3.4. We will investigate the attractive delta-well variant (see figure 4.1):

$$V(x) \doteq -D \sum_{n=-\infty}^{\infty} \delta(x - (n + \frac{1}{2})\, a) \quad , \qquad D > 0 \tag{4.11}$$

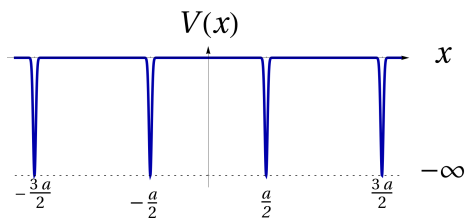

Figure 4.1: Sketch of the Kronig-Penney model with attractive delta peaks spaced by the lattice constant a .

The corresponding unshifted Hamiltonian

$$\mathbb{H}\,\psi(x) = E\,\psi(x) \quad ,$$

$$-\frac{d^2}{dx^2}\psi(x) - D \sum_{n=-\infty}^{\infty} \delta(x - (n + \frac{1}{2})\, a)\ \psi(x) = E\,\psi(x) \tag{4.12}$$

can be solved by the following ansatz in the nth interval:

$$\psi(x) = A_n\, e^{ik(x-n\,a)} + B_n\, e^{-ik(x-n\,a)} \quad , \tag{4.13}$$

with a momentum $k^2 = E$. The Bloch theorem (4.1) with $q \not\equiv k$ relates the amplitudes in the nth interval to the ones in the reference interval (here $n = 0$)

$$\forall n: \qquad A_n = A_0\, e^{iqna} \quad , \qquad B_n = B_0\, e^{iqna} \quad . \tag{4.14}$$

The constraints on the wave function are the same as for the single delta peak in section 3.4 . Rearranging the arising equations and bundling them in a matrix

equation we get

$$\underbrace{\begin{pmatrix} e^{ika} - e^{iqa} & e^{-ika} - e^{iqa} \\ ik\left(e^{iqa} - e^{ika}\right) - D\,e^{iqa} & ik\left(-e^{iqa} + e^{-ika}\right) - D\,e^{iqa} \end{pmatrix}}_{det \stackrel{!}{=} 0} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad . \tag{4.15}$$

Taking the determinant of the matrix and equating it with zero, we obtain the following transcendental equation for the wave momentum k and crystal momentum q

$$\cos(qa) = \cos(ak) + \frac{D}{2}\frac{\sin(ak)}{k} \quad . \tag{4.16}$$

This equation determines the dispersion relation for the periodic attractive delta-well potential. All waves with a momentum k and a Bloch momentum q complying with (4.16) are solutions of the Schrödinger equation. Usually, one plots the right-hand side as a function of k and solves for the band edges by investigating the intersections of the plot with ± 1, which are the limits of the left hand side cosine: the plot can be seen in figure 4.2.

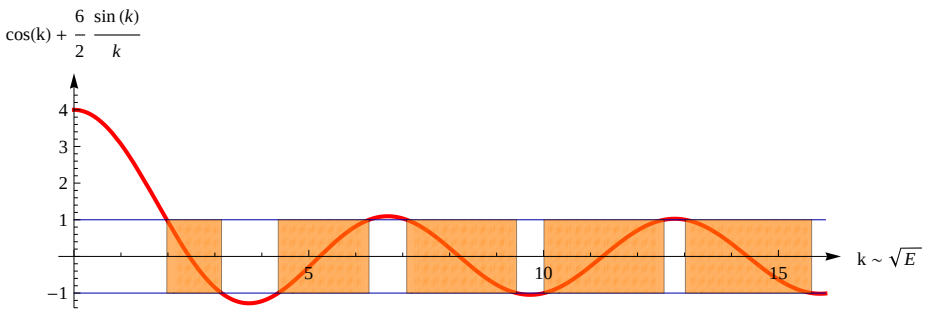

Figure 4.2: Transcendental dispersion relation (4.16) for the delta-well potential with the ansatz for piece-wise plane waves and $E > 0$ setting $a = 1,\ D = 6$. The orange areas mark the energy bands resp. allowed momenta k

Up to now, we only considered the possible wave functions with an energy $E > 0$ (in the shifted Hamiltonian $\mathbb{H}^{(1)}$). As a reminder, the single attractive delta-well had a bound state at a negative energy. This "bound" state also exists in the periodic case where it extends to a band as well. Here the energy becomes smaller than zero such that we consider "momenta" of $k = \sqrt{-E}$. The ansatz for the bound state solution in the nth interval then becomes

$$\psi(x) = A_n\, e^{k(x-n\,a)} + B_n\, e^{-k(x-n\,a)} \quad , \tag{4.17}$$

with the same relations for A_n and B_n as above in (4.14). Note that this would be the same ansatz as for $E > 0$ if we considered the former k which now would become imaginary. With similar calculations we retrieve the transcendental dispersion relation

$$\cos(qa) = \cosh(ak) - \frac{D}{2}\frac{\sinh(ak)}{k} \quad , \tag{4.18}$$

which is plotted in figure 4.3.

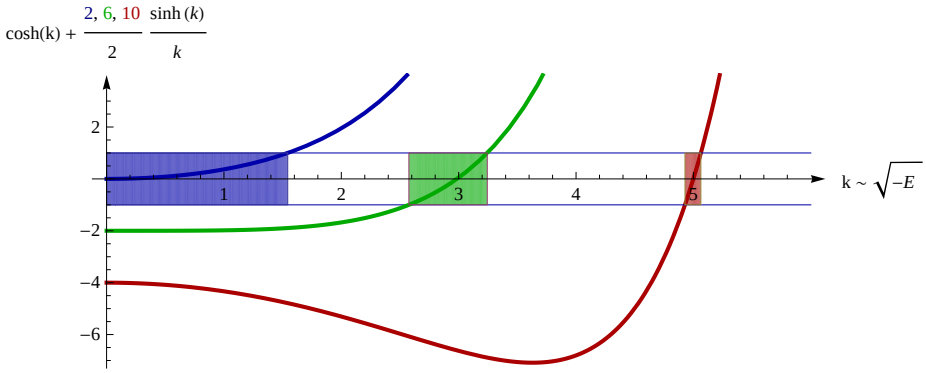

Figure 4.3: Transcendental dispersion relation (4.18) for the delta-well potential with the ansatz for the bound state and $E < 0$ setting $a = 1,\ D = 2, 6, 10$. The coloured areas mark the energy bands resp. allowed momenta k .

In the case when the delta-well strength obeys $0 < D \leq 2$, the energy band of the "bound" state reaches zero and is cut off at a certain threshold crystal momentum q. If the strength $D > 2$, then we have a full energy band with all valid Bloch momenta $-\frac{\pi}{a} \leq q \leq \frac{\pi}{a}$ and the whole band lies below $E = 0$.
The ground state energy E_0 is negative and marks the lower edge of the "bound" state band. Since the transcendental dispersion relations can only be solved numerically (and depend on the respective parameters a and D), we leave E_0 undetermined. In order to find the superpotential (2.8), we need the ground state wave function. With respect to the zero mode, we can simplify (4.17): due to symmetry reasons we can assume $A_0 = B_0$ and thus take a hyberbolic cosine as an ansatz:

$$\psi_0(x) = A\,\cosh(kx) = A\left(e^{kx} + e^{-kx}\right) \quad , \qquad k = \sqrt{-E_0} \quad , \tag{4.19}$$

for the shifted Hamiltonian $\mathbb{H}^{(1)}$

$$-\frac{d^2}{dx^2}\psi(x) - D\sum_{n=-\infty}^{\infty}\delta(x - (n + \frac{1}{2})\,a)\;\psi(x) - E_0\,\psi(x) = E\,\psi(x) \quad . \tag{4.20}$$

Now we can determine the superpotential:

$$W = -\frac{\psi_0'(x)}{\psi_0(x)} = -k\,\tanh(kx) = -\sqrt{-E_0}\,\tanh(\sqrt{-E_0}\,x) \quad . \tag{4.21}$$

Let us check whether we retrieve the constant potential in between the delta peaks:

$$\begin{aligned} V^{(1)}(x) &= (-E_0)\,\tanh^2(\sqrt{-E_0}\,x) - \left(-\frac{-E_0}{\cosh(\sqrt{-E_0}\,x)}\right) \\ &= -E_0\,\frac{\sinh^2(\sqrt{-E_0}\,x) + 1}{\cosh^2(\sqrt{-E_0}\,x)} \\ &= -E_0 \qquad (> 0) \quad . \end{aligned} \tag{4.22}$$

The delta peaks themselves are generated by the downward jump of the superpotential at $(n - \frac{1}{2})\,a,\ n \in \mathbb{Z}$ (see figure 4.4). The superpartner of the Kronig-Penney potential is

$$\begin{aligned} V^{(2)}(x) &= (-E_0)\,\tanh^2(\sqrt{-E_0}\,x) + \left(-\frac{-E_0}{\cosh(\sqrt{-E_0}\,x)}\right) \\ &= -E_0\,\frac{\cosh^2(\sqrt{-E_0}\,x) - 2}{\cosh^2(\sqrt{-E_0}\,x)} \\ &= -E_0\left(1 - \frac{2}{\cosh^2(\sqrt{-E_0}\,x)}\right) \quad . \end{aligned} \tag{4.23}$$

The sign of the superpotential's derivative changed for the superpartner and therefore the delta peaks go upwards now – just as in the single delta peak example from section 3.4. However, as derived in the introduction of this chapter, the superpartner has the same spectrum as the initial potential and the "bound" state band does not get lost: instead of a constant potential in between the delta peaks we now have sinks going down to zero. These sinks have the shape of the Pöschl-Teller potential $V(x) \propto \frac{1}{\cosh^2(x)}$ which is reflectionless (more information on this topic can be found in [12]). Thus, the sinks are attractive enough to contain the "bound" state energy band and even keep all the scattering properties of the potential, i.e. $R = 0$ for the constant potential, as we derived in section 2.7.

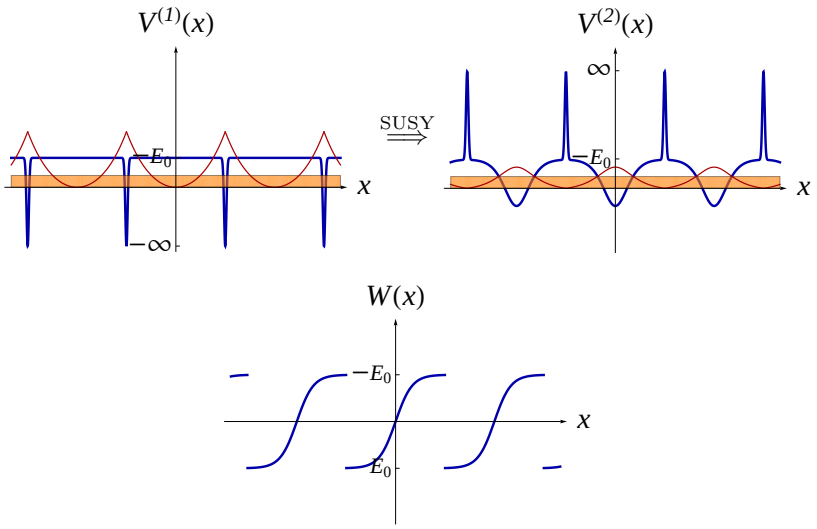

Figure 4.4: Top left: periodic attractive delta-well potential; top right: periodic delta-barrier potential with sinks in between; bottom: periodic superpotential with discontinuous jump generating the Dirac delta functions. Note: the delta peaks are sketched as narrow Gaussians.

Instead of guessing the ansatz for the zero mode, one can also start from the Riccati equation and integrate it:

$$
\begin{aligned}
& W^2(x) - \frac{d}{dx}W(x) = -E_0 \\
\Longrightarrow \quad & \frac{dW}{W^2 + E_0} = dx \\
\overset{\int}{\Longrightarrow} \quad & \frac{\arctan\left(\frac{W}{\sqrt{E_0}}\right)}{\sqrt{E_0}} = x - x_0 \\
\Longrightarrow \quad & W(x) = \sqrt{E_0}\ \tan\left(\sqrt{E_0}\,(x - x_0)\right) \\
\overset{x_0 \overset{!}{=} 0}{\Longrightarrow} \quad & W(x) = -\sqrt{-E_0}\ \tanh(\sqrt{-E_0}\,x) \quad , \qquad E_0 < 0 \quad .
\end{aligned}
\tag{4.24}
$$

This is exactly the same result as we obtained before by guessing the hyberbolic cosine ansatz for the zero mode. As one can clearly see, SUSY is exact and the periodic breaking condition (4.4) is fulfilled: the area under the superpotential compensates completely over one whole period. What is more, solving the Riccati equation for $V^{(2)}(x)$ to find its superpartner, one obviously retrieves $V^{(1)}(x)$ again ($\left(W^{(2)}(x)\right)^2 + \left(W^{(2)}(x)\right)' = V^{(1)}(x) \ \Rightarrow\ W^{(2)}(x) = -W^{(1)}(x)$). Also checking for the eigenvalue of the superpartner's zero mode $\psi_0^{(2)}(x) = \left(\cosh(kx)\right)^{-1}$ yields zero, hence the two potentials are completely isospectral (the derivation of the upper wave band is valid for both attractive and repulsive periodic delta peaks).

4.3 Lamé Potentials

The name Lamé potential originates from the corresponding Schrödinger equation which is known as the Lamé equation in the mathematics literature. In comparison to the delta peaks which describe point-like contact interactions, the Lamé potentials smoothly model a lattice with a finite attraction: they consist of the Jacobi elliptic functions which are briefly described in appendix B. An extensive discussion can be found in [10] resp. [5]. The Lamé potentials are defined on the real axis by

$$
V(x, m) = a\,(a+1)\,m\,\mathrm{sn}^2(x, m) \quad , \tag{4.25}
$$

where $\mathrm{sn}(x, m)$ is the Jacobi elliptic sine which depends on the real elliptic modulus parameter $0 \leq m \leq 1$. In case of $m = 0$ it becomes the usual trigonometric sine providing us with a rigid rotor potential, whereas in the case of $m = 1$ the Lamé potential reduces to the already mentioned Pöschl-Teller potential (see e.g. [12]). For a simpler notation, we will omit the modulus parameter m as an argument from now on. It is remarkable that the Lamé potential's energy band edges can be solved semi-analytically in contrast to most other periodic potentials as for instance the Kronig-Penney model in the last subsection. Hereby, for any integer value $a \in \mathbb{N}$ the Lamé potential has a bound bands and a continuum band on top.
In the following, we will investigate the supersymmetric partners for the first three integer values of $a = 1, 2, 3$.

4.3.1 $a = 1$

The Lamé potential with one bound band and the continuum state on top is defined as

$$V(x) \doteq 2m\,\mathrm{sn}^2(x) \quad . \tag{4.26}$$

The bound band edge energy values are given by

$$E_0 = m \qquad , \qquad E_1 = 1 \quad , \tag{4.27}$$

along with the corresponding unnormalized eigenfunctions

$$\psi_0(x) = \mathrm{dn}(x) \qquad , \qquad \psi_1(x) = \mathrm{cn}(x) \quad . \tag{4.28}$$

The continuum band starts at $E_2 = 1 + m$ and its eigenfunction is given by $\psi_2(x) = \mathrm{sn}(x)$. For the $m \to 0$ limit, (4.26) becomes a rigid rotor potential ($V(x) \to 2m\,\sin(x)$) and we correctly get the two eigenvalues $E_0 = 0$ and $E_1 = 1$. On the other hand, letting $m \to 1$ we obtain the Pöschl-Teller potential $V(x) \to 2 - \frac{2}{\cosh(x)}$ (cf. [3] and [12]) and the bound band shrinks to a single energy level at $E_0 = E_1 = 0$, whereas the continuum starts at the supremum of the potential $E_2 = 2$.
The superpotential (2.8) yields

$$W(x) = m\,\frac{\mathrm{sn}(x)\,\mathrm{cn}(x)}{\mathrm{dn}(x)} \quad . \tag{4.29}$$

After some lengthy algebraic rearrangements using the differentiation rules (B.4) and addition theorems (B.3), we indeed obtain the shifted initial potential

$$V^{(1)}(x) = W^2(x) - W'(x) = 2m\,\mathrm{sn}^2(x) - m \quad . \tag{4.30}$$

The partner potential (2.10) turns out to be

$$V^{(2)}(x) = 2 - m + \frac{2m-2}{\mathrm{dn}^2(x)} \quad .$$

Comparing with (B.5) shows that it can be written in terms of a translated sinus amplitudinis

$$V^{(2)}(x) = 2\left(1 - \mathrm{dn}^2(x+K)\right) - m = 2m\,\mathrm{sn}^2(x+K) - m \quad . \tag{4.31}$$

Now we clearly see that the two partner potentials describe the same physical potential shape only translated by a half-period as described at the end of section 4.1. Thus, we obtain nothing new, the Lamé potential with $a = 1$ is self-isospectral. In general, all higher Lamé potentials are just multiples of this example and one could suspect them all to be self-isospectral. Yet this assumption proves to be wrong: in fact, the derivative of the superpotential (being even for $a = 1$) is the reason for the self-reproduction as discussed in the previous chapters. However, this will change for higher integer values of a. Nevertheless, we see that $a = 1$ yields unbroken SUSY because the superpotential is odd and its area compensates exactly over one period of $2K$.

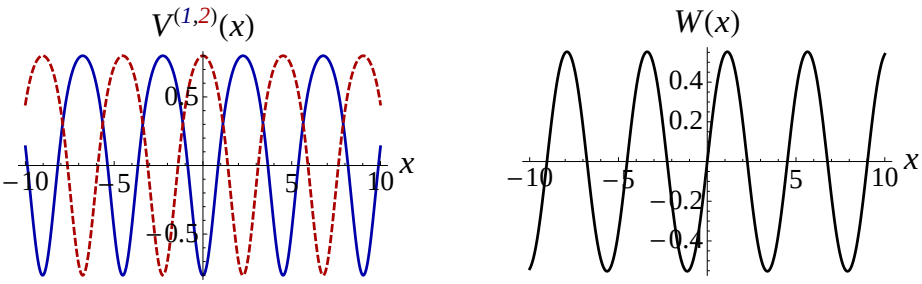

Figure 4.5: The Lamé potential with $a = 1$ reproduces itself under SUSY transformations. The left plot shows the partner potentials and the right one the superpotential for $m = 0.8$.

4.3.2 $a = 2$

Unlike in the last subsection, we will retrieve a new potential from the Lamé potential with two bound energy bands

$$V(x) \doteq 6m\,\mathrm{sn}^2(x) \quad . \tag{4.32}$$

Defining $\delta \doteq \sqrt{1 - m + m^2}$ and $B \doteq 1 + m + \delta$, we have band edges with the following properties:

E	$\psi^{(1)}$
$\left\{ \begin{array}{c} 0 \\ 3\delta - B \end{array} \right.$	$\left. \begin{array}{c} B - 3m\,\mathrm{sn}^2(x) \\ \mathrm{cn}(x)\,\mathrm{dn}(x) \end{array} \right\}$
$\left\{ \begin{array}{c} 2B - 3 \\ 2B - 3m \end{array} \right.$	$\left. \begin{array}{c} \mathrm{sn}(x)\,\mathrm{dn}(x) \\ \mathrm{sn}(x)\,\mathrm{cn}(x) \end{array} \right\}$
4δ	$B - 2\delta - 3m\,\mathrm{sn}^2(x)$

Table 1: Band edge properties of the Lamé potential with $a = 2$ and its partner potential.

Thus, the ground state reads

$$\psi_0(x) = 1 + m + \delta - 3m\,\mathrm{sn}^2(x) \quad , \tag{4.33}$$

and the superpotential yields

$$W(x) = 6m\,\frac{\mathrm{sn}(x)\,\mathrm{cn}(x)\,\mathrm{dn}(x)}{\psi_0(x)} \quad . \tag{4.34}$$

This looks already very different than the previous case. Checking back for the initial Lamé potential gives

$$V^{(1)}(x) = 6m\,\mathrm{sn}^2(x) - 2 - 2m + 2\delta \equiv V(x) - 2 - 2m - 2\delta \quad , \tag{4.35}$$

and the partner potential now becomes

$$V^{(2)}(x) = -\big(6m\,\mathrm{sn}^2(x) - 2 - 2m + 2\delta\big) + 72m^2\,\frac{\mathrm{sn}^2(x)\,\mathrm{cn}^2(x)\,\mathrm{dn}^2(x)}{\big(\psi_0(x)\big)^2} \quad . \tag{4.36}$$

This does not look like the supersymmetric partner of the previous section. Indeed, the partner potentials from integer $a \geq 2$ on do not satisfy (4.8) anymore and give rise to a new potential, the spectrum of which is completely analytically known from $\mathbb{H}^{(1)}$. The two partner potentials are plotted in figure 4.6: for $m = 0.8$ one can nicely see how the partner potential $V^{(2)}(x)$ is narrower but deeper than the initial Lamé potential and has another small sink at its top, such that it is intuitively plausible that the spectra actually are the same. Checking all the eigenfunctions of the partner system $\mathbb{H}^{(2)}$ (retrieved by $\psi_n^{(2)}(x) \propto \mathbb{A}\,\psi_n^{(1)}(x)$ as described in section 4.1), one thoroughly finds their eigenvalues to be the same as for $\mathbb{H}^{(1)}$. Also the partner potential's ground state wave function $\psi_0^{(2)}(x)$ is mapped to zero $\mathbb{A}^\dagger\psi_0^{(2)}(x) = 0$. Therefore, the two potentials both have their ground states at $E = 0$, or alternatively the equivalent condition (4.4) is indeed satisfied: SUSY is also exact for $a = 2$. In contrast to $a = 1$, one can see that the superpotential is now leaning to the right, thus it has an unsymmetric derivative which gives the two partner potentials their different shapes. The mechanism always stays the same for all the investigated potentials. The derivative of the superpotential determines how the superpartner will change from the initial potential.

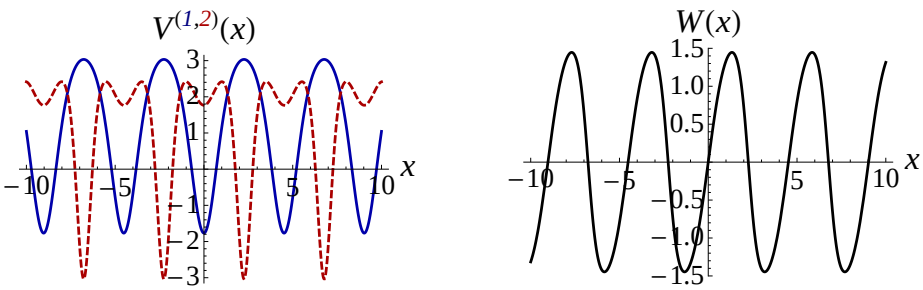

Figure 4.6: The Lamé potential with $a = 2$ gives rise to a new periodic potential. The left plot shows the partner potentials and the right one the superpotential for $m = 0.8$.

4.3.3 $a = 3$

Ultimately, we will take a look at the $a = 3$ case having three bound energy bands and the continuum which was depicted in the **plot on the front page**:

$$V(x) \doteq 12m\,\mathrm{sn}^2(x) \quad . \tag{4.37}$$

By defining $\delta_1 \doteq \sqrt{1-m+4m^2}$, $\delta_2 \doteq \sqrt{4-m+m^2}$ and $\delta_3 \doteq \sqrt{4-7m+4m^2}$, we get the following band edges:

E	$\psi^{(1)}$
0	$\mathrm{dn}(x)\left(1+2m+\delta_1-5m\,\mathrm{sn}^2(x)\right)$
$3-3m+2\delta_1-2\delta_2$	$\mathrm{cn}(x)\left(2+m+\delta_2-5m\,\mathrm{sn}^2(x)\right)$
$3+2\delta_1-2\delta_3$	$\mathrm{sn}(x)\left(2+2m+\delta_3-5m\,\mathrm{sn}^2(x)\right)$
$2-m+2\delta_1$	$\mathrm{sn}(x)\,\mathrm{cn}(x)\,\mathrm{dn}(x)$
$4\delta_1$	$\mathrm{dn}(x)\left(1+2m-\delta_1-5m\,\mathrm{sn}^2(x)\right)$
$3-3m+2\delta_1+2\delta_2$	$\mathrm{cn}(x)\left(2+m-\delta_2-5m\,\mathrm{sn}^2(x)\right)$
$3+2\delta_1+2\delta_3$	$\mathrm{sn}(x)\left(2+2m-\delta_3-5m\,\mathrm{sn}^2(x)\right)$

Table 2: Band edge properties of the Lamé potential with $a = 3$ and its partner potential.

The zero mode

$$\psi_0(x) = \mathrm{dn}(x)\left(1+2m+\delta_1-5m\mathrm{sn}^2(x)\right) \tag{4.38}$$

gives rise to the corresponding superpotential

$$m\,\frac{\mathrm{sn}(x)\,\mathrm{cn}(x)}{\mathrm{dn}(x)}\,\frac{11+2m+\delta_1-15m\,\mathrm{sn}^2(x)}{\psi_0(x)} \quad . \tag{4.39}$$

Checking back for the initial potential by (2.7) we get

$$V^{(1)}(x) = 12m\,\mathrm{sn}^2(x) - 2 - 5m + 2\delta_1 \quad , \tag{4.40}$$

and advancing to the partner potential yields

$$\begin{aligned} V^{(2)}(x) = &-\left(12m\,\mathrm{sn}^2(x) - 2 - 5m + 2\delta_1\right) \\ &+ 2m^2\left(\frac{\mathrm{sn}^2(x)\,\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)}\right)\left(\frac{11+2m+\delta_1-15m\,\mathrm{sn}^2(x)}{\psi_0(x)}\right)^2 \quad . \end{aligned} \tag{4.41}$$

Again, the partner potential $V^{(2)}(x)$ is a differently shaped potential which is strictly isospectral to $V^{(1)}(x)$ with the same band edges as given in the table 2. As illustrated in figure 4.7, the shape looks similar to the $a = 2$ case though.

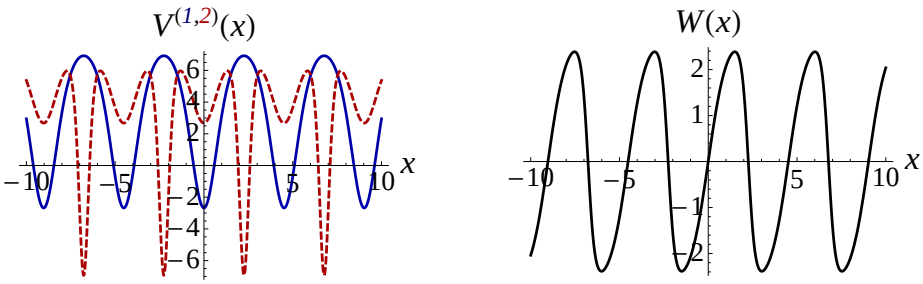

Figure 4.7: The Lamé potential with $a = 3$ also gives rise to a new periodic potential. The left plot shows the partner potentials and the right one the superpotential for $m = 0.8$.

The advance to higher integer values of a is straightforward.
In conclusion, the Lamé potentials with $a = 2, 3, ...$ bound bands are not self-isospectral, their SUSY partners are distinct new periodic potentials (although consisting of Jacobi elliptic functions as well) with strictly the same $2a + 1$ energy band eigenvalues. Moreover, SUSY appears to be exact for all $a \in \mathbb{N}$.

5 Conclusion

In this bachelor thesis, the basic supersymmetric quantum mechanics formalism using $N = 2$ SUSY (resp. the equivalent closed superalgebra $sl(1/1)$) in one dimension has been studied. We then examinedham a few relatively simple applications which provided a more intuitive insight into the mechanism of SUSY and finally we discussed the results of applying SUSY to periodic potentials.
In order to mathematically deal with SUSY, we presented the afore-mentioned SUSY-algebras, which have additional anticommutators in contrast to ordinary Lie algebras.
Generally, supersymmetry transfers bosonic states into fermionic ones whereas the Hamiltonian remains invariant: the energy spectrum becomes degenerate except for the bosonic ground state at $E = 0$ in the case of exact SUSY (as nicely seen for the single attractive delta-well which flips to a repulsive delta-barrier). If SUSY is broken, i.e. no ground state has an energy eigenvalue at $E = 0$, then the two spectra are completely equal. The limiting values of the superpotential, from which the two partner potentials are derived, essentially determine whether SUSY is exact or broken.
In the last chapter, we found that periodic potentials are completely isospectral in contrast to ordinary potentials. This was traced to the missing requirement of the zero mode's normalizability. Finally, we exploited the feature of SUSY to give rise to a new partner potential by enlarging the class of Lamé potentials.

6 Perspectives

Using SUSY, the hydrogen atom can be solved very nicely and quickly in comparison to the standard approach in quantum mechanics. In [8] and [1] one may find this topic briefly discussed.
With the coverage of one-dimensional problems in this thesis, one can now advance to multi-dimensional SUSY although the formalism changes slightly. In [1] this has been outlined for $d = 2$ dimensions, a nice application to two-dimensional Lamé potentials can be found in [13].
As already indicated in the first and second chapter, the SUSY algebra can easily be extended to another pair of SUSY operators $\mathbb{Q}_2$ and $\mathbb{Q}_2^\dagger$. One therefore obtains a supersymmetric Hamiltonian covering three quantum systems with a three-fold degenerate spectrum. The Hamiltonians can then be factorized in two different ways $\mathbb{H}^{(1)} = \{\mathbb{Q}_1,\ \mathbb{Q}_1^\dagger\} = \{\mathbb{Q}_2,\ \mathbb{Q}_2^\dagger\}$. This second-order SUSY transformation can give rise to another new potential – take the periodic Lamé potentials for instance. With the supercharges defined in [2] one admittedly does not find a distinct new periodic potential for the cases of $a = 1, 2$, but in the case of $a = 3$ we get a third potential being isospectral to the other two partner potentials we found in section 4.3.3. The three potentials are plotted for $m = 0.8$ in figure 6.1.

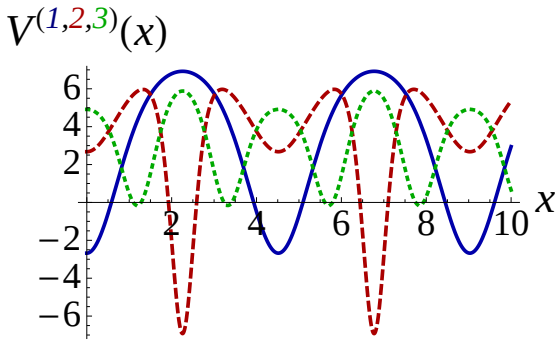

Figure 6.1: The Lamé potential with $a = 3$ has two partner potentials as discussed in [2] – the plot was made with an elliptic modulus of $m = 0.8$.

In view of all these results from SUSY quantum mechanics, one may indeed conclude that it not only helps to understand the SUSY breaking mechanism as it was examined in the beginning, but also provides interesting new analytically solvable potentials with already known spectra from their initial partner potentials.

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Adrian Oeftiger

A Comparison of Unit Systems

During this bachelor thesis, we used two different unit systems: the SI units and modified natural units such that $\hbar = 2m \equiv 1$. Here we will give a short list of the essential relations in both unit systems:

object	SI units	modified natural units $\hbar = 2m \equiv 1$
Hamiltonian	$\mathbb{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$	$\mathbb{H} = -\frac{d^2}{dx^2} + V$
initial potential	$V^{(1)}(x) = \frac{\hbar^2}{2m}\frac{\psi_0''}{\psi_0}$	$V^{(1)}(x) = \frac{\psi_0''}{\psi_0}$
SUSY operator	$\mathbb{A} = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$	$\mathbb{A} = \frac{d}{dx} + W(x)$
Riccati equation	$V^{(1)}(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x)$	$V^{(1)}(x) = W^2(x) - W'(x)$
super potential	$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'}{\psi_0}$	$W(x) = -\frac{\psi_0'}{\psi_0}$
partner potential	$V^{(2)}(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x)$	$V^{(2)}(x) = W^2(x) + W^(x)$
unbroken zero mode	$\psi_0(x) = \exp\left(\pm\frac{\sqrt{2m}}{\hbar}\int_{-\infty}^{x} dy\ W(y)\right)$	$\psi_0(x) = \exp\left(\pm\int_{-\infty}^{x} dy\ W(y)\right)$

B Jacobi Elliptic Functions

Here we compile a few important facts about the Jacobi elliptic functions (cf. among others [5]).

$$\text{real elliptic modulus :} \qquad 0 \leq m \leq 1 \tag{B.1}$$

$$\text{real elliptic quarter period :} \qquad K(m) = \int_0^{\frac{\pi}{2}} d\theta \, \frac{1}{\sqrt{1 - m \, \sin^2 \theta}} \tag{B.2}$$

Elementary Functions, Periods and Special Cases

name of function	symbol	real period	$m = 0$	$m = 1$
Sinus Amplitudinis	$\mathrm{sn}(x, m)$	$4K(m)$	$\sin(x)$	$\tanh(x)$
Cosinus Amplitudinis	$\mathrm{cn}(x, m)$	$4K(m)$	$\cos(x)$	$\frac{1}{\cosh(x)}$
Delta Amplitudinis	$\mathrm{dn}(x, m)$	$2K(m)$	1	$\frac{1}{\cosh(x)}$

Relations Between Jacobi Elliptic Functions

$$m \, \mathrm{sn}^2(x, m) = m - m \, \mathrm{cn}^2(x, m) = 1 - \mathrm{dn}^2(x) \tag{B.3}$$

Derivatives of Jacobi Elliptic Functions

$$\begin{aligned} \frac{d}{dx} \, \mathrm{sn}(x, m) &= \mathrm{cn}(x, m) \, \mathrm{dn}(x, m) \\ \frac{d}{dx} \, \mathrm{cn}(x, m) &= -\mathrm{sn}(x, m) \, \mathrm{dn}(x, m) \\ \frac{d}{dx} \, \mathrm{dn}(x, m) &= -m \, \mathrm{sn}(x, m) \, \mathrm{cn}(x, m) \end{aligned} \tag{B.4}$$

Translations by Quarter Periods

$$\begin{aligned} \mathrm{sn}(x + K(m), m) &= \frac{\mathrm{cn}(x, m)}{\mathrm{dn}(x, m)} \\ \mathrm{cn}(x + K(m), m) &= -\sqrt{1 - m} \, \frac{\mathrm{sn}(x, m)}{\mathrm{dn}(x, m)} \\ \mathrm{dn}(x + K(m), m) &= \sqrt{1 - m} \, \frac{1}{\mathrm{dn}(x, m)} \end{aligned} \tag{B.5}$$